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The spectrum of HSOLSSOM(h^n) where h is even☆

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Abstract

In this paper, we describe a generalized product construction and a construction using Steiner pentagon systems to obtain holey self-orthogonal Latin squares with symmetric orthogonal mates (HSOLSSOM). We investigate the existence of HSOLSSOM(h^n) for even h. We first improve the known result for $h = 2$ and show that a HSOLSSOM(2^n) exists for all $n \geqslant 5$ except possibly for $n \in E = \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$. As a consequence, we then establish that for $h \equiv 2 \pmod 4$ and $h \geqslant 10$ a HSOLSSOM(h^n) exists for all $n \geqslant 5$ except possibly for $n \in E$. More conclusively, we show that for $h \equiv 0 \pmod 4$, a HSOLSSOM(h^n) exists if and only if $n \geqslant 5$. We are also able to apply our results to construct a unipotent SOLSSOM(62), the existence of which was previously unknown.

1. Introduction

A *quasigroup* is an ordered pair $(Q, \cdot)$, where Q is a set and $(\cdot)$ is a binary operation on Q such that the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

are uniquely solvable for every pair of elements a, b in Q. It is well known (e.g., see [3]) that the multiplication table of a quasigroup defines a *Latin square*; that is, a Latin square can be viewed as the multiplication table of a quasigroup with the headline and sideline removed. For a finite set Q, the *order* of the quasigroup $(Q, \cdot)$ is $|Q|$. A quasigroup $(Q, \cdot)$ is called *idempotent* if the identity

$$x^2 = x$$

holds for all x in Q.

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Two quasigroups $(Q,\cdot)$ and $(Q,*)$ defined on the same set Q are said to be *orthogonal* if the pair of equations $x\cdot y=a$ and $x*y=b$, where a and b are any two given elements of Q, are satisfied simultaneously by a unique pair of elements from Q. We remark that when two quasigroups are orthogonal, then their corresponding Latin squares are also orthogonal in the usual sense.

Let S be a set and $\boldsymbol{H}=\{S_1,S_2,\ldots,S_k\}$ be a set of disjoint subsets of S. A *holey Latin square* having *hole set* $\boldsymbol{H}$ is an $|S|\times|S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains an element of S or is empty,

(2) every element of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i\times S_i$ are empty for $1\leqslant i\leqslant n$ (these subarrays are referred to as *holes*),

(4) element $s\in S$ occurs in row or column t if and only if $(s,t)\in(S\times S)\backslash\bigcup_{1\leqslant i\leqslant n}(S_i\times S_i)$. The *order* of L is $|S|$. Two holey Latin squares on symbol set S and hole set $\boldsymbol{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S\times S)\backslash\bigcup_{1\leqslant i\leqslant n}(S_i\times S_i)$. We shall use the notation IMOLS$(s;s_1,\ldots,s_n)$ to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\boldsymbol{H}=\{S_1,S_2,\ldots,S_n\}$, where $s=|S|$ and $s_i=|S_i|$ for $1\leqslant i\leqslant n$. If $\boldsymbol{H}=\emptyset$, we obtain a MOLS(s). If $\boldsymbol{H}=\{S_1\}$, we simply write IMOLS(s,s_1) for the orthogonal pair of holey Latin squares.

If $\boldsymbol{H}=\{S_1,S_2,\ldots,S_n\}$ is a partition of S, then a holey Latin square is called a *partitioned incomplete Latin square*, denoted by PILS. The *type* of the PILS is defined to be the multiset $\{|S_i|:\ 1\leqslant i\leqslant n\}$. We shall use an "exponential" notation to describe types: so that $t_1^{u_1}\ldots t_k^{u_k}$ denotes u_i occurrences of $t_i, 1\leqslant i\leqslant k$, in the multiset. Two orthogonal PILS of type T will be denoted by HMOLS(T).

A holey Latin square is called *self-orthogonal* if it is orthogonal to its transpose. For self-orthogonal holey Latin squares we use the notation SOLS(s), ISOLS(s,s_1) and HSOLS(T) for the case of $\boldsymbol{H}=\emptyset,\{S_1\}$ and a partition $\{S_1,S_2,\ldots,S_n\}$, respectively.

If any two PILS in a set of t PILS of type T are orthogonal, then we denote the set by t HMOLS(T). Similarly, we may define t MOLS(s) and t IMOLS(s,s_1).

A *holey SOLSSOM* having partition $\boldsymbol{P}$ is 3 HMOLS (having partition $\boldsymbol{P}$), say A,B,C where $B=A^T$ and $C=C^T$. Here a SOLSSOM stands for a *self-orthogonal Latin square* (SOLS) with a *symmetric orthogonal mate* (SOM). A holey SOLSSOM of type T will be denoted by HSOLSSOM(T).

HSOLSSOMs have been useful in the construction of resolvable orthogonal arrays invariant under the Klein 4-group [6], Steiner pentagon systems [8] and three-fold BIBDs with block size seven [17]. The existence of a HSOLSSOM(h^n) has been investigated for $h=2$ and for h odd. It is easy to see that $n\geqslant 5$ is a necessary condition for the existence of such a design. The following existence results are known.

Theorem 1.1 (Mullin and Stinson [10], Zhu [15]). *A* HSOLSSOM(2^n) *exists for all odd* $n\geqslant 5$ *except possibly* $n\in\{15,33,87\}$ *and for all even* $n\geqslant 46$ *except possibly* $n\in\{48,50,52,54,58,62,68,72,74,76,80,84,88,108,114\}$.

Theorem 1.2 (Mullin and Zhu [11]). *If h is an odd integer, then a* HSOLSSOM(h^n) *exists if and only if $n \geqslant 5$ is odd except possibly for $h = 3$ and $n \in \{11, 13, 15, 17, 19, 23, 27, 33, 39, 51, 59, 87\}$.*

We wish to remark that $n = 15, 33$ can be now removed in Theorem 1.2. In this paper, we describe a generalized product construction and a construction using Steiner pentagon systems to obtain holey self-orthogonal Latin squares with symmetric orthogonal mates. We investigate the existence of HSOLSSOM(h^n) for even h. We first improve the known result for $h = 2$ and show that a HSOLSSOM(2^n) exists for all $n \geqslant 5$ except possibly for $n \in E = \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$. As a consequence, we then establish that for $h \equiv 2 \pmod 4$ and $h \geqslant 10$ a HSOLSSOM(h^n) exists for all $n \geqslant 5$ except possibly for $n \in E$. More conclusively, we show that for $h \equiv 0 \pmod 4$, a HSOLSSOM(h^n) exists if and only if $n \geqslant 5$.

2. Constructions

2.1. A construction using holey Steiner pentagon systems

Let K_n be the complete undirected graph with n vertices. A *pentagon system* (PS) of order n is a pair $(K_n, \boldsymbol{B})$, where $\boldsymbol{B}$ is a collection of edge disjoint pentagons which partition the edges of K_n. A *Steiner pentagon system* (SPS) of order n is a pentagon system $(K_n, \boldsymbol{B})$ with the additional property that every pair of vertices are joined by a path of length 2 in exactly one pentagon of $\boldsymbol{B}$.

Let Q be an n-set and let K_n be based on Q. It is well known [7] that a quasigroup $(Q, \cdot)$ satisfying the three identities $x^2 = x, (yx)x = y$ and $x(yx) = y(xy)$ is equivalent to a SPS $(K_n, \boldsymbol{B})$. Here a pentagon $(x, y, z, u, v) \in \boldsymbol{B}$ if and only if $xy = z$ and $yx = v$ for $x \neq y$ and $x^2 = x$ for all $x \in Q$. A quasigroup associated with a SPS is called a *Steiner pentagon quasigroup* (briefly denoted by SPQ).

A *partitioned incomplete quasigroup* (PIQ) is a partial quasigroup whose multiplication table with the headline and sideline removed is a PILS. The type of the PIQ is the type of its associated PILS. A PIQ of type h^n satisfying the identities $(yx)x = y$ and $x(yx) = y(xy)$ is denoted by HSPQ(h^n).

A *holey Steiner pentagon system* of type h^n (HSPS(h^n)) is a SPS with n disjoint holes of equal-size h. A HSPS(h^n) is essentially equivalent to a HSPQ(h^n).

Theorem 2.1.1. *Suppose there exists a holey Steiner pentagon system of type h^n. Then there exists a* HSOLSSOM(h^n).

Proof. Let $(Q, \cdot)$ be a HSPQ(h^n) associated with the given HSPS(h^n). Then $(Q, \cdot)$ satisfies both the identities

$$(yx)x = y \tag{1}$$

and

$$x(yx) = y(xy). \tag{2}$$

First of all, we shall prove that the holey quasigroup $(Q,\cdot)$ is self-orthogonal. Suppose $(Q,*)$ is the transpose of $(Q,\cdot)$. Then for given elements a and b in Q, we must essentially show that the equations $xy = a$ and $x * y = b$ are uniquely solvable. In fact, (1) implies

$$bx = y \tag{3}$$

since the equation $x * y = b$ is equivalent to $yx = b$. From (2) and (3), we have

$$b(xb) = x(bx) = xy.$$

So, the equation $xy = a$ implies

$$b(xb) = a.$$

Since $(Q,\cdot)$ is a quasigroup, xb and then x can be determined uniquely. From (3), we further get a unique solution for y. This shows that $(Q,\cdot)$ is self-orthogonal.

Let us now define a holey quasigroup $(Q,\circ)$ where $x \circ y = x(yx)$ and the products are defined by the operation $(\cdot)$. It is easy to see from (2) that $x \circ y = y \circ x$, so the quasigroup $(Q,\circ)$ is commutative. Next, we shall show that $(Q,\circ)$ and $(Q,\cdot)$ are orthogonal.

Suppose $xy = c$ and $x \circ y = d$ where c and d are given elements from Q. Then we have $d = x(yx)$ and $y(xy) = yc$. We further get $d = yc$ from (2). Since $y = (yc)c$ from (1), we know that $y = dc$. Again from (1), we get $x = (xy)y = c(dc)$. It can be readily verified that these solutions for x and y satisfy the initial equations. Note that $xy = (c(dc))(dc) = c$ follows from (1). By definition, $x \circ y = x(yx) = y(xy) = (dc)((c(dc))(dc))$, and from (1) it follows that $x \circ y = (dc)c = d$. This shows that $(Q,\circ)$ and $(Q,\cdot)$ are orthogonal and the proof of the theorem is complete. □

Lemma 2.1.2. *There exist a* HSPS(2^6) *and a* HSOLSSOM(2^6).

Proof. A HSPQ(2^6) has been constructed by computer search [9]. Let $Q=\{1,2,\dots,12\}$ and let $\{\{1,7\},\{2,6\},\{3,5\},\{4,9\},\{8,10\},\{11,12\}\}$ be the hole set of the HSPS. Then the HSPS(2^6) contains the following 12 pentagons:

$$(1,2,5,4,12),\quad (1,3,12,9,10),\quad (1,4,3,6,8),\quad (1,5,11,10,6),$$

$$(1,9,2,8,11),\quad (2,3,8,7,12),\quad (2,4,10,3,11),\quad (2,7,9,5,10),$$

$$(3,7,6,11,9),\quad (4,6,12,10,7),\quad (4,8,5,7,11),\quad (5,6,9,8,12).$$

Using the constructions in the proof of Theorem 2.1.1, we obtain a HSOLSSOM(2^6) shown in Figs. 1 and 2.

	5	12	3	11	10		6	2	9	8	4
12		8	10	4		9	11	1	5	3	7
10	12		1		8	6	7	11	4	2	9
8	11	6		2	12	10	5		3	7	1
6	1		12		9	11	4	7	2	10	8
5		4	7	12		3	1	8	11	9	10
	10	9	6	8	11		3	5	12	4	2
4	9	2	11	7	3	12		6		1	5
11	8	7		10	5	2	12		1	6	3
3	7	11	2	9	1	4		12		5	6
9	4	10	8	1	7	5	2	3	6		
2	3	1	5	6	4	8	9	10	7		

Fig. 1. A HSOLS(2^6).

	4	9	6	10	11		3	8	12	2	5
4		7	3	12		5	1	11	9	10	8
9	7		8		1	11	12	6	2	4	10
6	3	8		1	10	12	7		11	5	2
10	12		1		8	4	11	2	7	6	9
11		1	10	8		9	4	12	5	3	7
	5	11	12	4	9		2	10	6	8	3
3	1	12	7	11	4	2		5		9	6
8	11	6		2	12	10	5		3	7	1
12	9	2	11	7	5	6		3		1	4
2	10	4	5	6	3	8	9	7	1		
5	8	10	2	9	7	3	6	1	4		

Fig. 2. A symmetric orthogonal mate.

2.2. *A generalized product construction*

In this section, we shall generalize the product type constructions of Lemmas 1 and 2 in [15] to allow more flexible input designs in constructing HSOLSSOMs.

We denote by ILS(s,s_1) a holey Latin square of order s when it contains only one hole of size s_1. An element in the hole of an ILS is said to be *evenly distributed* if it does not appear on the main diagonal and if when it appears in one cell, then it must appear also in its symmetric cell. If each element in the hole is evenly distributed, then we say that the ILS is *balanced.* Given 3 IMOLS, if one of the three ILS is balanced and if also each element in the hole determines $s - s_1$ distinct entries above the main diagonal in the other two squares, then we say that the 3 IMOLS are *compatible.*

A set of cells in a Latin square, which is based on a set S, is called a *transversal* if they intersect each row and each column exactly once and contain each element in

S exactly once. Two transversals are said to form a *symmetric pair* if when a cell appears in one transversal, then its symmetric cell appears in the other transversal.

Lemma 2.2.1. *Suppose q is an odd prime power, $q \geqslant 7$. Suppose there exist 3 MOLS(m) and compatible 3 IMOLS$(m+e_t, e_t)$ where m is even, $t = 1, 2, \ldots, \frac{1}{2}(q-5)$, $k = \sum_{1 \leqslant t \leqslant (q-5)/2}(2e_t)$. Then there exists a HSOLSSOM of type $m^{(q-1)}(m+k)^1$.*

Proof. Let $L_\lambda = (a_{ij})$, where $a_{ij} = a_i + \lambda a_j$, a_i, a_j, $\lambda \in \mathrm{GF}(q) = \{a_0, a_1, \ldots, a_{q-1}\}$ such that $a_0 = 0$. Let α be a primitive element of GF(q). It is easy to see that the Latin squares $L_1, L_{\alpha^1}, L_{\alpha^2}, \ldots, L_{\alpha^{q-2}}$ form $q-1$ MOLS(q) and that $L_{\alpha^1}, L_{\alpha^2}, \ldots, L_{\alpha^d}$ are all self-orthogonal, where $d = \frac{1}{2}(q-3)$. The cells with entry 0 in L_{α^t} determine a common transversal, say tth transversal, of L_{α^d} and L_1 forming a SOLSSOM(q), where $t = 1, 2, \ldots, \frac{1}{2}(q-5)$. The transpose of the tth transversal is also a common transversal of the SOLSSOM(q), say (t^*)th transversal, which is determined by the cells with entry 0 in $(L_{\alpha^t})^T$. Let the diagonal be the 0th common transversal, then we know that all these transversals intersect in cell $(0,0)$ and are disjoint elsewhere. Now we present the following generalized product construction.

Begin with the SOLSSOM(q), where L_{α^d} is the SOLS and L_1 is the SOM. Replace each cell of the SOLSSOM with an $m \times m$ array labelled by the element in that cell. If the cell is on the diagonal, then the array is empty. Suppose we are given M_1, M_2, M_3 to be 3 MOLS(m) based on an m-set M. For a cell above the diagonal and not in any of the $d-1$ symmetric pairs of transversals, the array consists of M_1 for the SOLS and M_3 for the SOM, while at the same time the array for its symmetric cell will consist of M_2^T and M_3^T. Suppose we are given N_{t1}, N_{t2}, N_{t3} to be compatible 3 IMOLS$(m+e_t, e_t)$ based on set $M \cup E_t$, where $|E_t| = e_t$ and N_{t3} is a balanced ILS. Suppose M_{tj}, A_{tj} and B_{tj} are the upper left part, the right part and the lower part of the N_{tj}, respectively, as shown below:

$$N_{tj} = \begin{array}{|c|c|} \hline M_{tj} & A_{tj} \\ \hline B_{tj} & \\ \hline \end{array}$$

Replace E_t by a disjoint e_t-set E_t^*, we obtain similarly compatible 3 IMOLS$(m + e_t, e_t)$ $N_{t1}^*, N_{t2}^*, N_{t3}^*$, where N_{t3}^* is a balanced ILS and M_{tj}^*, A_{tj}^* and B_{tj}^* are the upper left part, the right part and the lower part of the N_{tj}^*, respectively, as shown below:

$$N_{tj}^* = \begin{array}{|c|c|} \hline M_{tj}^* & A_{tj}^* \\ \hline B_{tj}^* & \\ \hline \end{array}$$

For any non-$(0,0)$ cell in the tth transversal, the array consists of M_{t1} for the SOLS and M_{t3} for the SOM, while at the same time the array for its symmetric cell will consist of $(M_{t2}^*)^T$ and $(M_{t3}^*)^T$. We suppose that the elements in E_t and E_t^* remain unchanged when labelling. Then we obtain the upper left part of the desired HSOLSSOM, where there are q empty subarrays each of size m down the diagonal. As shown in Fig. 3, the four corners form a hole of size $m+k$. What we need is to describe the right part and the lower part of the HSOLSSOM.

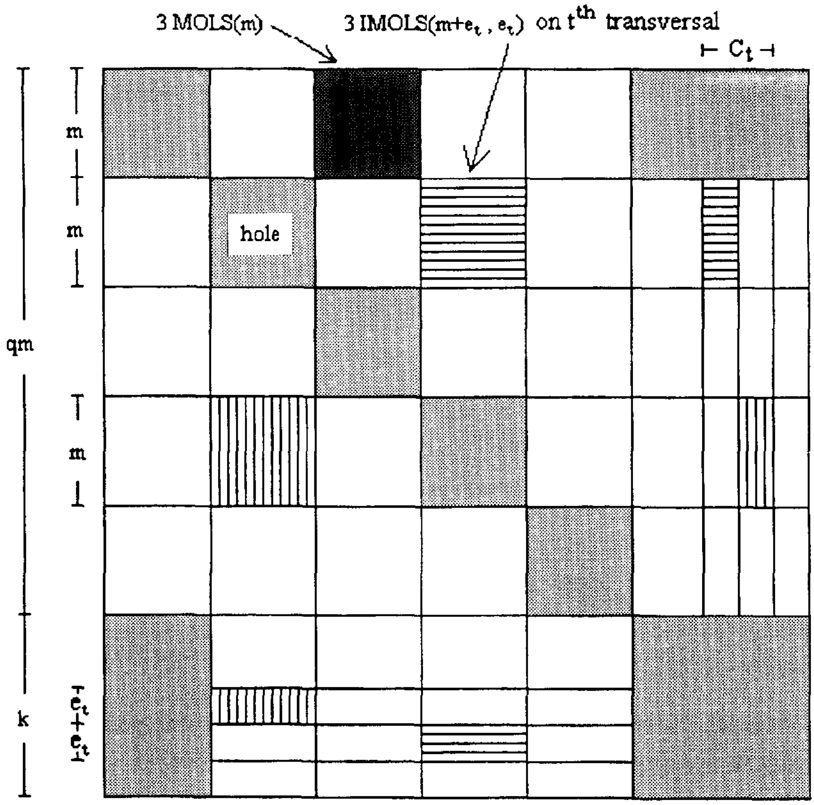

Fig. 3. HSOLSSOM $(m^{(q-1)}(m+k)^1)$.

The right part consists of the columns $C_1,\ldots,C_t,\ldots,C_{d-1}$ where column C_t comes from tth and (t^*)th transversals of the SOLSSOM(q). First, project the transversals to form two columns such that tth transversal is to the left and (t^*)th transversal to the right. Next, use the entries as labelling and replace each cell with an $m\times e_t$ array: A_{t1} for the tth transversal and $(B_{t2}^*)^T$ for the (t^*)th transversal in the SOLS, A_{t3} for the tth transversal and $(B_{t3}^*)^T$ for the (t^*)th transversal in the SOM. The lower part consists of the rows $R_1,\ldots,R_t,\ldots,R_{d-1}$ where row R_t comes from tth and (t^*)th transversals of the SOLSSOM(q). First, project the transversals to form two rows such that tth transversal is below and (t^*)th transversal is above. Next, use the entries as labelling and replace each cell with an $e_t\times m$ array: B_{t1} for the tth transversal and $(A_{t2}^*)^T$ for the (t^*)th transversal in the SOLS, B_{t3} for the tth transversal and $(A_{t3})^T$ for the (t^*)th transversal in the SOM.

Now we obtain a holey self-orthogonal Latin square with an orthogonal mate which is almost symmetric. The only problem is that in the orthogonal mate some positions occupied by an element x in E_t have their symmetric positions occupied by another element y in E_t^*. Since N_{t3} and N_{t3}^* are both balanced, we may make the following adjustment so that the orthogonal mate becomes symmetric. We keep the entries x in

N_{t3} and y in N_{t3}^{*} unchanged if they appear above the diagonal, but interchange x and y if they appear below the diagonal.

It is a routine matter to see that the final squares are the desired HSOLSSOM of type $m^{(q-1)}(m+k)^{1}$. This completes the proof. □

If we are given A, B and C to be 3 IMOLS(v,n) such that $B^{T} = A$ and $C^{T} = C$, then we say that A and C form an ISOLSSOM(v,n). When m is even and $e = 0$ or e is odd, the existence of an ISOLSSOM$(m+e,e)$ implies the existence of compatible 3 IMOLS$(m+e,e)$. Therefore, we have the following corollary which is Lemma 2.1 in [16].

Corollary 2.2.2. *Suppose q is an odd prime power, $q \geqslant 7$. Suppose there exist* SOLSSOM(m) *and* ISOLSSOM$(m+e_t,e_t)$ *where m is even, $e_t = 0$ or e_t* odd > 0, $t = 1,2,\ldots,\frac{1}{2}(q-5)$, $k = \sum_{1\leqslant t\leqslant (q-5)/2}(2e_t)$. *Then there exists a* HSOLSSOM *of type $m^{(q-1)}(m+k)^{1}$.*

Lemma 2.2.3. *Suppose $q \geqslant 5$, q is an odd prime power or $q \equiv \pm 1 (\mathrm{mod}\, 6)$. Suppose there exist compatible* 3 IMOLS$(m+e_t,e_t)$ *where m is even, $t = 1,2,\ldots,\frac{1}{2}(q-1)$, $k = \sum_{1\leqslant t\leqslant (q-1)/2}(2e_t)$. Then there exists a* HSOLSSOM *of type $m^{q}k^{1}$.*

Proof. Suppose q is an odd prime power. Instead of the symmetric pairs of transversals intersecting in cell $(0,0)$ in the proof of Lemma 2.2.1, this time we get from L_{-1} the symmetric pairs of transversals in a SOLSSOM$(q) L_{\xi}$ and L_1, $\xi \in \mathrm{GF}(q)\backslash\{0,1,-1\}$. These transversals are disjoint with one another. The similar construction gives a HSOLSSOM of type $m^{q}k^{1}$. When $q \equiv \pm 1 (\mathrm{mod}\, 6)$, let $L_{\xi} = (a_{ij})$, $a_{ij} = i + \xi j$ in Z_q. L_2 and L_1 form a SOLSSOM(q) with an extra orthogonal mate L_{-1}, which determines the symmetric pairs of common transversals in L_2 and L_1. The remainder of the proof is the same as the prime power case. The proof is complete. □

The following corollary is Lemma 2.2 in [16]

Corollary 2.2.4. *Suppose $q \geqslant 5$, q is an odd prime power or $q \equiv \pm 1$ (mod 6). Suppose there exist* ISOLSSOM$(m+e_t,e_t)$ *where m is even, $e_t = 0$ or e_t odd > 0, $t = 1,2,\ldots,\frac{1}{2}(q-1)$, $k = \sum_{1\leqslant t\leqslant (q-1)/2}(2e_t)$. Then there exists a* HSOLSSOM *of type $m^{q}k^{1}$.*

2.3. Other constructions

In this section, we shall describe several other recursive constructions. The first one is simple but useful.

Construction 2.3.1 (*Filling in holes*). (1) Suppose there exists a HSOLSSOM of type $\{s_i : 1 \leqslant i \leqslant n\}$. Let $a \geqslant 0$ be an integer. For each i, $1 \leqslant i \leqslant n-1$, if there exists a

HSOLSSOM of type $\{s_{ij} : 1 \leqslant j \leqslant k(i)\} \cup \{a\}$, where $s_i = \sum_{1 \leqslant j \leqslant k(i)} s_{ij}$, then there is a HSOLSSOM of type $\{s_{ij} : 1 \leqslant j \leqslant k(i), 1 \leqslant i \leqslant n-1\} \cup \{a + s_n\}$.

(2) Suppose there exists a HSOLSSOM of type $\{s_i : 1 \leqslant i \leqslant n\}$. Suppose there exists also a HSOLSSOM of type $\{t_j : 1 \leqslant j \leqslant k\}$, where $s_n = \sum_{1 \leqslant j \leqslant k} t_j$. Then there is a HSOLSSOM of type $\{s_i : 1 \leqslant i \leqslant n-1\} \cup \{t_j : 1 \leqslant j \leqslant k\}$.

The next recursive construction for HSOLSSOM uses group divisible designs. A *group divisible design* (or GDD), is a triple $(X, \boldsymbol{G}, \boldsymbol{B})$ which satisfies the following properties:

(1) $\boldsymbol{G}$ is a partition of a set X (of *points*) into subsets called *groups*,

(2) $\boldsymbol{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{|G| : G \in \boldsymbol{G}\}$. A GDD $(X, \boldsymbol{G}, \boldsymbol{B})$ will be referred to as a K-GDD if $|B| \in K$ for every block B in $\boldsymbol{B}$. A TD(k,n) is a GDD of group type n^k and block size k. An RTD(k,n) is a TD(k,n) where the blocks can be partitioned into parallel classes. It is well known that the existence of an RTD(k,n) is equivalent to the existence of a TD$(k+1,n)$ or equivalently $(k-1)$ MOLS(n). We wish to remark that a special GDD with all groups of size one is essentially a pairwise balanced design (PBD), denoted by $(X, \boldsymbol{B})$. We use [1] as our standard design theory reference.

The following PBD construction is essentially [11, Lemma 3.1].

Construction 2.3.2. Suppose there exists a PBD $(X, \boldsymbol{B})$ and for each block $B \in \boldsymbol{B}$ there exists a HSOLSSOM$(h^{|B|})$. Then there exists a HSOLSSOM$(h^{|X|})$.

More generally, we can apply Wilson's fundamental construction for GDDs [14] to obtain a similar construction for HSOLSSOM.

Construction 2.3.3 (*Weighting*). Suppose $(X, \boldsymbol{G}, \boldsymbol{B})$ is a GDD and let $w : X \to \boldsymbol{Z}^+ \cup \{0\}$. Suppose there exists a HSOLSSOM of type $\{w(x) : x \in B\}$ for every $B \in \boldsymbol{B}$. Then there exists a HSOLSSOM of type $\{\sum_{x \in G} w(x) : G \in \boldsymbol{G}\}$.

The following product construction is essentially Lemma 3.4 in [10].

Construction 2.3.4. Suppose there exists a HSOLSSOM of type h^n. Let $m \geqslant 4$ and $m \neq 6, 10$. Then there exists a HSOLSSOM of type $(mh)^n$.

To apply the above constructions the following known result is useful.

Theorem 2.3.5 (Dénes and Keedwell [3]). *For any prime power p, there exists a* TD(k, p), *where* $3 \leqslant k \leqslant p + 1$.

Table 1
HSOLSSOM(2^n)

t	a	b	n
5	1		26
5	5		30
7	1	0	36
7	5	0	40
7	7	0	42
7	6	5	46
7	7	6	48
9	5	0	50
9	7	0	52
9	9	0	54
9	7	6	58
11	7	0	62
11	7	6	68
13	7	0	72
13	9	0	74
13	11	0	76
13	9	6	80
19	7	6	108
19	19	0	114

3. HSOLSSOMs of type 2^n

In this section, we shall improve the known result in Theorem 1.1 and show that a HSOLSSOM(2^n) exists for all $n \geqslant 5$ except possibly for $n \in E = \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$. As a consequence, we then establish that for $h \equiv 2 \pmod 4$ and $h \geqslant 10$ a HSOLSSOM(h^n) exists for all $n \geqslant 5$ except possibly for $n \in E$.

Lemma 3.1. *There exists a* HSOLSSOM *of type 2^n for $n \in \{26, 30, 36, 40, 42, 46, 48, 50, 52, 54, 58, 62, 68, 72, 74, 76, 80, 108, 114\}$.*

Proof. We start with a TD$(7, t)$ for $t \in \{7, 9, 11, 13, 19\}$. Delete $t - a$ points from one group and delete $t - b$ points from another group. We obtain a GDD of group type $t^5 a^1 b^1$ with block sizes 5, 6 and 7. Apply Weighting Construction and give weight two to each point of the GDD. We obtain a HSOLSSOM of type $(2t)^5(2a)^1(2b)^1$. For parameters of t, a and b shown in Table 1, there are HSOLSSOMs of type $2^t, 2^a, 2^b$ from Theorem 1.1 and Lemma 2.1.2. Therefore, we may apply Filling in Holes Construction to obtain a HSOLSSOM of type 2^n, where $n = 5t + a + b$. If we start with a TD$(6, t)$ and delete $5 - a$ points from one group, then the similar construction gives a HSOLSSOM of type 2^n where $n = 5t + a$. □

Lemma 3.2. *There exist compatible* 3 IMOLS(v, n) *for* $(v, n) = (10, 2)$ *and* $(k, 1)$, *where k is any odd integer greater than* 3.

6	9	8	1	7	5	2	0	3	4
2	7	9	8	1	4	6	3	0	5
8	3	4	9	0	2	5	7	1	6
9	8	0	5	4	1	3	6	2	7
4	6	1	3	8	7	0	9	5	2
0	5	7	2	9	8	4	1	6	3
3	1	6	4	2	9	8	5	7	0
5	0	2	7	6	3	9	8	4	1
1	2	3	0	5	6	7	4		
7	4	5	6	3	0	1	2		

8	1	7	9	0	3	5	4	2	6
9	8	2	4	5	1	0	6	3	7
5	9	8	3	7	6	2	1	0	4
0	6	9	8	2	4	7	3	1	5
3	2	6	5	1	9	8	7	4	0
6	0	3	7	4	2	9	8	5	1
4	7	1	0	8	5	3	9	6	2
1	5	4	2	9	8	6	0	7	3
2	3	0	1	6	7	4	5		
7	4	5	6	3	0	1	2		

0	5	4	2	6	9	7	8	1	3
3	1	6	5	8	7	9	4	2	0
6	0	2	7	5	8	4	9	3	1
4	7	1	3	9	6	8	5	0	2
1	8	0	9	2	4	6	3	7	5
9	2	8	1	0	3	5	7	4	6
2	9	3	8	4	1	0	6	5	7
8	3	9	0	7	5	2	1	6	4
5	6	7	4	1	2	3	0		
7	4	5	6	3	0	1	2		

Fig. 4. Compatible 3 IMOLS(10, 2).

Proof. For any odd integer $k \geqslant 5$ there exists a SOLSSOM(k) from [6,13,15]. Then there exist an ISOLSSOM(k, 1) and compatible 3 IMOLS(k, 1). An example of compatible 3 IMOLS(10, 2) is taken from [2] with some modifications and shown in Fig. 4, where the third square is a balanced ILS. □

Lemma 3.3. *There exists a* HSOLSSOM *of type* 2^n *for* $n = 33$ *and* 34.

Proof. Apply Lemma 2.2.3 with $q = 7$, $m = 8$, $e_1 = 1$, $e_2 = 2$, and $e_3 = 1$ or 2. Since there exist compatible 3 IMOLS($8 + e, e$) for $e = 1, 2$ from Lemma 3.2, we obtain a HSOLSSOM of type $8^7 8^1$, or type $8^7 10^1$. Further apply Construction 2.3.1 (1) with $a = 2$ and use HSOLSSOMs of type 2^5 and type 2^6 as input designs, which all exist from Theorem 1.1 and Lemma 2.1.2. We obtain the desired HSOLSSOMs of type 2^{33} and type 2^{34}. □

Lemma 3.4. *There exists a* HSOLSSOM *of type* 2^{38}.

Proof. Apply Corollary 2.2.2 with $q = 9$, $m = 8$, $e_1 = 0$ and $e_2 = 1$. We obtain a HSOLSSOM of type $8^8 10^1$. Further apply Construction 2.3.1 (1) with $a = 2$ and use HSOLSSOMs of types 2^5 and 2^6 as input designs. We obtain a HSOLSSOM(2^{38}). □

Lemma 3.5. *There exists a* HSOLSSOM *of type* 2^{44}.

Proof. Start with a TD(7, 7) which exists from Lemma 2.3.5. It is not difficult to find 5 points in the TD such that they intersect each block and each group at no more than two points. We then obtain a PBD with 44 points and block size 5, 6 or 7. Apply Construction 2.3.2. with $h = 2$. Since HSOLSSOMs of types $2^5, 2^6$ and 2^7 exist, we obtain the desired HSOLSSOM of type 2^{44}. □

Lemma 3.6. *There exists a* HSOLSSOM *of type* 2^n *for* $n = 84, 87$ *and* 88.

Proof. From [5] we have a GDD of group type 7^{13} with all blocks of size 7. Deleting one group, we get a PBD with 84 points and block sizes 7 and 6. If we delete two points from the GDD or delete from three groups of the GDD one point each such that they are not in the same block, then we get a PBD with 87 or 88 points and block sizes 7, 6 and 5. Apply Construction 2.3.2 with $h = 2$. Since HSOLSSOMs of types $2^5, 2^6$ and 2^7 exist, we obtain the desired HSOLSSOMs. □

Combining the results obtained in this section and the known result in Theorem 1.1, we may summarize with the following theorem.

Theorem 3.7. *There exists a* HSOLSSOM *of type* 2^n *for all* $n \geqslant 5$ *except possibly for* $n \in \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$

Applying Construction 2.3.4 with $h = 2$ and $m \geqslant 5$ odd, we obtain the following theorem from the results of Theorem 3.7.

Theorem 3.8. *Suppose* $h \equiv 2 \pmod 4$ *and* $h \geqslant 10$. *Then there exists a* HSOLSSOM *of type* h^n *for all* $n \geqslant 5$ *except possibly for* $n \in \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$.

4. HSOLSSOMs of type h^n where $h \equiv 0 \pmod 4$

In this section we shall make use of PBDs with block sizes from the set $H^4 = \{k : k \equiv 1 (\mathrm{mod}\, 4)$. This set has been investigated quite extensively by various authors, but the main tool for our constructions comes from [12]. If $(X, \boldsymbol{B})$ is a PBD with block sizes from the set H^4 and there is a point $x \in X$ which is contained exclusively in blocks of size 5, then we say that x is a *5-head* of $(X, \boldsymbol{B})$ and write $v = |X| \in B(5^\wedge, H^4)$. Evidently, $v \in B(5^\wedge, H^4)$ if and only if there exists an H^4-GDD of type $4^{(v-1)/4}$.

The following two lemmas are contained in [12, Theorem 2.10 and Lemma 3.4].

Lemma 4.1. $B(5^\wedge, 9) \supseteq \{v : v \equiv 1 (\mathrm{mod}\, 4), v \geqslant 21, v \neq 29, 33\} \backslash \{49, 73, 93, 113, 133, 153, 173, 193\}$.

Lemma 4.2. $B(5^\wedge, 9, 13) \supseteq \{v : v \equiv 1 (\mathrm{mod}\, 4), v \geqslant 21, v \neq 29, 33\} \backslash \{49\}$.

From Lemmas 4.1 and 4.2, we readily obtain the following two useful GDDs.

Lemma 4.3. *If* $n \geqslant 5$ *and* $n \neq 7, 8, 12$, *then there exists a* $\{5, 9, 13\}$-GDD *of type* 4^n.

Lemma 4.4. *If* $n \geqslant 5$ *and* $n \neq 7, 8, 12, 18, 23, 28, 33, 38, 43$, *and* 48, *then there exists a* $\{5, 9\}$-GDD *of type* 4^n.

Lemma 4.5. *There exists a* HSOLSSOM(4^n) *for all* $n \geqslant 5$.

Proof. From Theorem 1.2 and Lemma 4.3, we can give each point weight one in the $\{5, 9, 13\}$-GDD of type 4^n to obtain a HSOLSSOM of type 4^n for all $n \geqslant 5$, where $n \neq 7, 8, 12$. For $n = 7$, we apply Corollary 2.2.2 with $m = 4$, $q = 7$ and $e_1 = 0$. For $n = 8$ and 12, we apply Corollary 2.2.4 and $m = 4$, $q = 7, 11$ respectively, and $e_1 = 0$ or 1 to obtain HSOLSSOMs of type $4^7 4^1$ and $4^{11} 4^1$. This completes the proof. □

Lemma 4.6. *There exists a* HSOLSSOM(8^n) *for all* $n \geqslant 5$.

Proof. From Theorem 3.7 and Lemma 4.3, we can give each point weight two in the $\{5, 9.13\}$-GDD of type 4^n to obtain a HSOLSSOM(8^n) for all $n \geqslant 5$, where $n \neq 7, 8, 12$. For $n = 7$, we apply Corollary 2.2.2 with $m = 8$, $q = 7$ and $e_1 = 0$. For $n = 8$, a HSOLSSOM of type $8^7 8^1$ has already been constructed in the proof of Lemma 3.4. For $n = 12$, we apply Corollary 2.2.4 with $m = 8$, $q = 11$ and $e_t = 0$ or 1 to get a HSOLSSOM of type $8^{11} 8^1$ and this completes the proof. □

Lemma 4.7. *If* $h \equiv 0 \pmod 4$ *and* $h \geqslant 16$, *then there exists a* HSOLSSOM(h^n) *for all* $n \geqslant 5$, *where* $h \neq 24$.

Proof. We apply Construction 2.3.4 to the results of Lemmas 4.5 and 4.6. □

Lemma 4.8. *If there exist a* RTD$(6, m)$ *and a* HSOLSSOM(2^m), *then there exists a* HSOLSSOM(12^n) *where* $n = m$ *and* $m + 1$.

Proof. To a RTD$(6, m)$ we may adjoin 0 or 6 infinite points so as to form $\{6, 7, m\}$-GDDs of type 6^m and 6^{m+1}, using either one parallel class of blocks or one parallel class of blocks together with the infinite points as groups. Using the fact that we have HSOLSSOM(2^n) for $n = 6$ and 7 from Theorem 3.7, we may give each point of the resulting GDD weight two to obtain the desired result. □

Lemma 4.9. *There exists a* HSOLSSOM(12^n) *for* $n \in \{7, 8, 12, 18, 23, 28, 33, 38, 43, 48\}$.

Proof. We apply Lemma 4.8 by choosing $m \in \{7, 11, 17, 23, 27, 32, 37, 43, 47\}$. It is easy to verify that all the necessary conditions are satisfied. □

Lemma 4.10. *There exists a* HSOLSSOM(12^n) *for all* $n \geqslant 5$.

Proof. If $n \geqslant 5$ and $n \notin \{7,8,12,18,23,28,33,38,43,48\}$, then from Theorem 1.2, we can give each point weight three in a $\{5,9\}$-GDD of type 4^n, which exists from Lemma 4.4. This gives a HSOLSSOM(12^n) and the result follows from Lemma 4.9. □

Lemma 4.11. *If there exists a* RTD$(6,m)$, *then there exists a* HSOLSSOM(24^n), *where* $n = m$ *and* $m+1$.

Proof. As in the proof of Lemma 4.8, we first form $\{6,7,m\}$-GDDs of type 6^m and 6^{m+1}. We know that a HSOLSSOM(4^m) exists from Lemma 4.5, so we can give each point of the GDDs weight 4 to obtain HSOLSSOM(24^n), where $n = m$ and $m+1$. □

Lemma 4.12. *There exists a* HSOLSSOM(24^n) *for* $n \in \{8,10,12,14,15,16,18,20,22,24,28,32\}$.

Proof. If $n \in \{8,10,12,14,16,18,20,24,28,32\}$, we can apply Lemma 4.11 by choosing $m \in \{7,9,11,13,16,17,19,23,27,32\}$ to obtain a HSOLSSOM(24^n). For $n = 15$, we can delete one point from a 7-GDD of type 7^{13} (see [5]) to obtain a 7-GDD of type 6^{15}. We give weight 4 to each point of this GDD to obtain a HSOLSSOM(24^{15}), using HSOLSSOM(4^7) from Lemma 4.5. For $n = 22$, we start with a RTD$(6,23)$ and delete one block from a particular parallel class so as to form a $\{5,6,22\}$-GDD of type 6^{22}. Using HSOLSSOM(4^n) for $n = 5, 6$, and 22 from Lemma 4.5, we can give weight 4 to this GDD and obtain a HSOLSSOM(24^{22}). □

Lemma 4.13. *There exists a* HSOLSSOM(24^n) *for all* $n \geqslant 5$.

Proof. If $n \notin \{8,10,12,14,15,16,18,20,22,24,28,32\}$, then we know that there is a HSOLSSOM(2^n) from Theorem 3.7. We can then apply Construction 2.3.4 with $m = 12$ and $h = 2$ to get a HSOLSSOM(24^n). Combining this with the result of Lemma 4.12, we complete the proof of the lemma. □

The results of Lemmas 4.5, 4.6, 4.7, 4.10 and 4.13 can be combined to give the main result of this section.

Theorem 4.14. *If* $h \equiv 0 \pmod 4$, *then there exists a* HSOLSSOM(h^n) *if and only if* $n \geqslant 5$.

5. Concluding remarks

The problem of existence of HSOLSSOM(h^n) for even h has been completely settled for $h \equiv 0 \pmod 4$. For $h \equiv 2 \pmod 4$, we have obtained fairly conclusive results in Theorems 3.7 and 3.8, except for the case $h = 6$ which remains under investigation.

It is perhaps worth mentioning that the HSOLSSOM(10^6) from Theorem 3.8 can be used to obtain a unipotent SOLSSOM(62), the existence of which was previously unknown. The construction involves filling in the holes, using ISOLSSOM(12, 2) (see [4]) and SOLSSOM(12). We therefore obtain the following improvement to the spectrum of SOLSSOM(v) (see, for example, [15]):

Theorem 5.1. *A SOLSSOM(v) exists for all positive integers v, with the exception of $v = 2, 3, 6$ and the possible exception of $v = 10, 14, 46, 54, 58, 66, 70$.*

Note added: Since this paper was submitted for publication, several new HSOLSSOMs have been found. In particular, the possible exceptions n = 8, 10, 15, 16, 20 can now be removed from Theorems 3.7 and 3.8. The possible exceptions v = 46, 54, 58 can also be removed from Theorem 5.1.

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